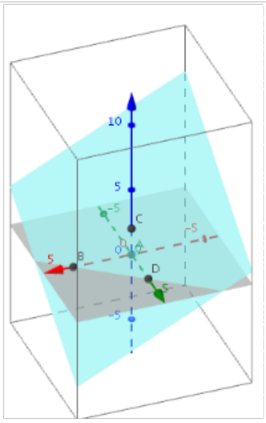
1. Use Geogebra to graph (in 3D) the plane 3x+4y+6z = 12.
2. Algebraically find the x, y, and z intercepts of the plane.
3. Let A, B, and C be the x, y, and z intercepts you found in (2). Find $\vec{AB}$ and $\vec{AC}$.
4. Evaluate $\vec{AB} \times \vec{AC}$. How does this relate to the equation of the plane?



$3x+4y+6z=12$

x-intercept : $y=z=0$

$3x=12 \rightarrow x=4$ $\quad (4,0,0)=A$

y-int : $x=z=0$

$4y=12 \rightarrow y=3$ $\quad (0,3,0)=B$

z-int : $x=y=0$

$6z=12 \rightarrow z=2$ $\quad (0,0,2)=C$

$\vec{AB} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}, \quad \vec{AC} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$

$\vec{AB} \times \vec{AC} \quad \begin{vmatrix} i & j & k \\ -4 & 3 & 0 \\ -4 & 0 & 2 \end{vmatrix} = 6\vec{i} + 8\vec{j} + 12\vec{k}$

this vector is $\perp$ to $\vec{AB}$ and $\vec{AC}$

components resemble the equation

Vectors that are 'in' the plane are parallel to the plane
The vector 3i + 4j+6k is perpendicular to the plane 3x+4y+6z = 12.

We defined the plane using two vectors here, and those vectors came from points that were in the plane.

If we have three points in a plane, we can find a find a vector that is perpendicular to the vectors defined by those points.

In this example, we found a vector that was perpendicular to AB and AC.

Make a conjecture: If I tell you that -2i + 3j + 5k is a vector perpendicular to a plane, what equation could could you write for that plane?

-2x+3y+5z = something

Let's see if we can prove this to be true.

R A

Let's say we know the points (x,y,z) and (x1,y1,z1) are in a plane. Let's also say that we know the vector <a,b,c> is perpendicular to this plane.

$\vec{n}$

If $\vec{AR} \perp \vec{n}$, then $\vec{AR} \cdot \vec{n} = 0$

$\vec{AR} \cdot \vec{n} = |\vec{AR}||\vec{n}| \cos\theta$

$$\vec{AR} = \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix}$$

$$\vec{AR} \cdot \vec{n} = a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$ax + by + cz = ax_1 + by_1 + cz_1$$

To be clear, we are saying that A = (x1,y1,z1) is a point that we know the coordinates for. Point R is just a general set of coordinates (x,y,z) in the plane.

Suppose I tell you that <3,-1,6> is a vector perpendicular to a plane containing the point (1,2,3). Write an equation for this plane.

x_1, y_1, z_1

$$3x - y + 6z = 3(1) - 1(2) + 6(3)$$

$$3x - y + 6z = 19$$

Here, the point we know (point A) is (1,2,3).

A couple definitions now:

- A vector that is perpendicular to a plane is called a <u>normal vector</u>.
- The form of equation we are using here is called the <u>Cartesian form of an equation for a plane</u>:

ax + by+cz = constant

Let's mix things up a bit.

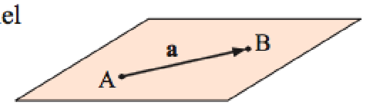
We've said that a single normal vector can describe a plane. We didn't start with that vector in our warm-up.

We can also describe a plane by two non-parallel vectors located within it.

From our textbook:

A vector is parallel to a plane if, by travelling along the vector, we remain on the plane.

For example, if A and B both lie on a plane then $\mathbf{a} = \overrightarrow{AB}$ is parallel to the plane.



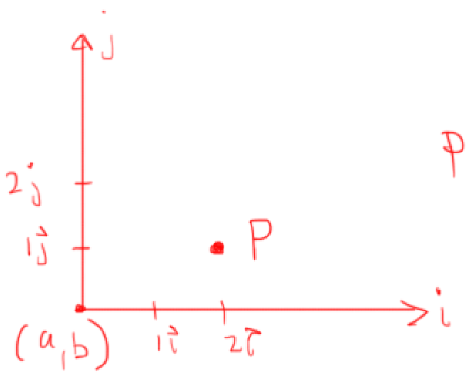
In our warm up example, we had two vectors in the plane that were parallel to the plane, but not parallel to each other.

We showed this by finding the cross product:

$$\vec{AB} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}, \quad \vec{AC} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \longrightarrow \text{if } \vec{AB} \parallel \vec{AC}, \text{ then } |\vec{AB} \times \vec{AC}| = 0$$

The cross product is not the zero vector, so AB is not parallel to AC.

We've used the idea that two non-parallel vectors can map out a plane for a long time.



P has coordinates

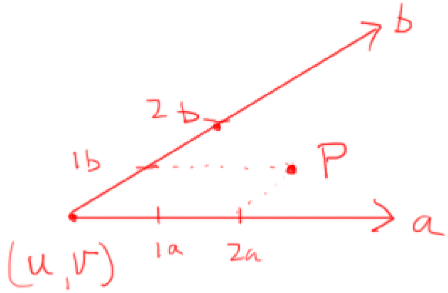
$(a + 2\vec{i},\ b + 1\vec{j})$

(0,0)

$$\vec{P} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In our coordinate system, relative to (a,b), point P has coordinates (2,1).

We can also do this for vectors that are not perpendicular:



$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

(0,0)

$$P = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

In our weird coordinate system, P has coordinates (2,1).

This idea extends to 3D as well.

Let's use AB and AC as our non-perpendicular vectors in the plane. Let's use point A as our starting point. Back to the warm-up:

point A

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$$

$$\text{starting point} + \lambda \cdot \overrightarrow{AB} + \mu \cdot \overrightarrow{AC}$$

(λ, μ) is like the coordinates relative to point A

This form is called the <u>vector form</u> of the equation of a plane.

An example from Cirrito:

EXAMPLE 27.9

Find the vector equation of the plane containing the vectors $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\0\\-1\end{pmatrix}$ which also includes the point $(1, 2, 0)$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-1\end{pmatrix}$$

...and from Haese and Harris:

Example 20 Self Tutor

Find the equation of the plane through $A(-1, 2, 0)$, $B(3, 1, 1)$, and $C(1, 0, 3)$:

a in vector form b in Cartesian form.

vector from origin to

a) starting point : $\begin{pmatrix}-1\\2\\0\end{pmatrix}$, two

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\3\end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$$

b) point in plane: $(-1, 2, 0)$, normal vector from $\vec{AB}$ and $\vec{AC}$

$$\begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ 2 & -2 & 3 \end{vmatrix} = i(-1) - 10j - 6k$$

$$= -i - 10j - 6k$$

equation: $-x - 10y - 6z = -1(-1) - 10(2) - 6(0)$

$$-x - 10y - 6z = -19$$

$$x + 10y + 6z = 19$$

Switching from vector form to Cartesian form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$$

equal vectors have equal components

$$x = 4 - 4\lambda - 4\mu$$

$$y = 3\lambda \longrightarrow \lambda = \frac{y}{3}$$

$$z = 2\mu \longrightarrow \mu = \frac{z}{2}$$

$$x = 4 - 4\left(\frac{y}{3}\right) - 4\left(\frac{z}{2}\right)$$

$$6x = 6\cdot 4 - 6\cdot 4\left(\frac{y}{3}\right) - 6\cdot 4\left(\frac{z}{2}\right)$$

$$6x = 24 - 8y - 12z$$

$$6x + 8y + 12z = 24$$

$$3x + 4y + 6z = 12$$

This should make us feel good because this was the plane equation we started with.

So cool!